ON THE CLASSIFICATION OF G-GRADED TWISTED ALGEBRAS

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Abstract. Let G denote a group and let W be an algebra over a commutative ring R. We will say that W is a G-graded twisted algebra (not necessarily commutative, neither associative) if there exists a G-grading $W = \oplus_{g \in G} W_g$ where each summand W_g is a free rank one R -module, and W has no monomial zero divisors (for each pair of nonzero elements w_a, w_b en W_a and W_b their product is not zero, $w_a w_b \neq 0$). It is also assumed that W has an identity element.

In this article, methods of group cohomology are used to study the general problem of classification under graded isomorphisms. We give a full description of these algebras in the associative cases, for complex and real algebras. In the nonassociative case, an analogous result is obtained under a symmetry condition of the corresponding associative function of the algebra, and when the group providing the grading is finite cyclic.

1. Introduction

G-graded twisted algebras were introduced in [5], and independently in [14], as distinguished mathematical structures which arise naturally in theoretical physics [15], [16], [17], [18], [19], and [6]. By one of these algebras we mean the following: Let G denote a group. An R-algebra W (not necessarily commutative, neither associative) will be called a *G-graded twisted algebra* if there exists a G-grading, i.e., $W = \oplus_{g \in G} W_g$, with $W_a W_b \subset W_{ab}$, in which each summand W_g is an R -module of free rank one. We assume that W has an identity element $1 \in W_e$, where W_e denotes the graded component corresponding to the identity element e of G. We also require that W has no *monomial zero divisors*, i.e., for each pair of nonzero elements $w_a \in W_a$, and $w_b \in W_b$, their product must be non zero, $w_a w_b \neq 0$ (for a general study of nonassociative algebras, the reader may consult [9].)

Besides its interest for physicists, these algebras are natural objects of study for mathematicians, since they are related to generalizations of Lie algebras that include transformation parameters with noncommutative and/or nonassociative properties, as defined in [4], where generalizations of some results of Scheunert [10] on epsilon or color (super) Lie algebras are discussed. One important case, that of G-graded twisted *division* algebras (in which every nonzero element has a left and a right inverse, but where the left and right inverses do not necessarily coincide) already includes the classical

quaternion and octonion division algebras. They are a new remarkable class of (noncommutative and nonassociative) division algebras over the reals. Their classification is addressed in [11], [12].

For the general case of G-graded twisted algebras over the complex numbers, first attempts towards their classification appear in [3], where rudimentary techniques of group cohomology were introduced. Those methods are exploited in this article to obtain necessary and sufficient conditions for two algebras to be isomorphic under graded isomorphisms (see Theorem 4 and its corollaries). On the other hand, classical techniques of group representation theory readily give a complete classification in the associative case (Corollaries 1 and 3.) This last result is achieved via Theorem 1, which allows to represent these algebras as quotients of certain group rings. A similar description is possible in the general, not associative case, where instead of group rings the corresponding structures are quotients of *loop rings,* objects, notwithstanding, poorly studied in the literature.

As for general nonassociative algebras, we deal in this article only with the simplest cases. We study algebras over the complex and real numbers which are graded by finite abelian groups, and whose associativity function –see (2.1) below– satisfies certain symmetry condition (Theorems 7 and 8.)

2. Definitions and basic concepts

Definition 1. Let G denote a group and let W be an algebra over a commutative ring R. We will say that W is a *G-graded twisted algebra* (not necessarily commutative neither associative) if there exists a G-grading $W = \oplus_{g\in G} W_g$, with $W_a W_b \subset W_{ab}$, where each summand W_g is a free rank one R -module. It is required that W has no monomial zero divisors. This condition means that for each pair of nonzero elements $w_a \in W_a$ and $w_b \in W_b$, $w_a w_b \neq 0$. It is also assumed that W has an identity element $1 = w_e \in W_e$, where e denotes the identity element of G.

Since each graded component W_g is a free rank one R -module, we may choose $w_g \in W_g$ such that $B = \{w_g : g \in G\}$ is a basis for W as an R-module. For any such fixed basis B we may associate a function (the *structure constant with respect to* B) $C_B : G \times G \to R^*$, which takes values in $R^* = R - \{0\}$, and such that for any pair of elements $w_a \in W_a$, $w_b \in W_b$, $w_a w_b = C_B(a, b) w_{ab}$. We notice that $w_a w_e = C_B(a, e) w_a$, implies $C_B(a, e) = 1$, for all a in G. The fact that W has no monomial zero divisors implies that C_B must take values in a a subdomain A of the multiplicative group R^*. If we are interested in stressing this fact, we will write $C_B : G \times G \to A$. On the other hand, if B is understood, we will simply write $C : G \times G \to A$, omitting the subscript.

If R is a field, and $A \subset R$, a subfield, the commutative and associative properties of W can be understood by means of the following two functions,

$q : G \times G \to A$, and $r : G \times G \times G \to A$, defined as:

$$\begin{aligned} q(a,b) &= C(a,b)C(b,a)^{-1} \\ r(a,b,c) &= C(b,c)C(ab,c)^{-1}C(a,bc)C(a,b)^{-1} \end{aligned} \tag{2.1}$$

If G is abelian, it holds that $w_a(w_b w_c) = r(a,b,c)(w_a w_b)w_c$; and that $w_a w_b = q(a,b)w_b w_a$.

Definition 2. By a *morphism* between two G-graded twisted algebras $W = \oplus_{g\in G} W_g$ and $V = \oplus_{g\in G} V_g$ we mean an unitarian homomorphism of R-algebras $\phi : W \to V$. If it also preserves the grading, i.e., $\phi(W_g) \subset V_g$, we say the morphism is graded.

Remark 1. Let us fix $B_1 = \{w_g : g \in G\}$ and $B_2 = \{v_g : g \in G\}$ bases for W and V, respectively. Defining a *graded morphism* $\phi : W \to V$ amounts to giving a function $\varphi : G \to R$ such that $\phi(w_g) = \varphi(g)v_g$. Clearly, ϕ will be an isomorphism if and only if for all g, $\varphi(g)$ is a unit in R.

The classification problem of G-graded twisted algebras admits at least two versions.

(1) W and V can be isomorphic as *graded* algebras, that is, there are *graded* isomorphisms $\phi : W \to V$ and $\psi : V \to W$ such that $\phi\psi$ and $\psi\phi$ is the identity.
(2) W and V can be isomorphic as R-algebras, without taking into consideration the grading.

In this article *we will restrict to the case where R is a field*, that we will denote by k. *It will also be assumed that G is a finite group.*

3. G-graded twisted algebras and Loop rings

Let W be a G-graded twisted algebra, and let $C : G \times G \to A \subset k^*$ be the structure constant with respect to a fixed basis B. The function C gives rise to an extension E of A by G, that will be denoted by $A \times_C G$, defined as follows: $E_W = \{(\alpha, g) : \alpha \in A, g \in G\} \subset A \times G$. We endow this set with the following operation:

$$(\alpha, a).(\beta, b) = (\alpha\beta C(a,b), ab)\,.$$

It is easy to see that whenever W is associative, E_W is a group, actually an extension of A by G. In general, if W is not associative, then E_W turns out to be a *loop*, a structure which is almost a group, except that its binary operation is not necessarily associative, but where there is still an identity element, and where each element has a right and a left inverse, not necessarily equal. As in the case of an ordinary group, *the loop ring $R = k[E_W]$* may be defined analogously.

The structure of any associative G-graded twisted algebra can be understood using the following fundamental result.

Theorem 1. *Let $W = \oplus_{g\in G} W_g$, and let $C : G\times G \to A$ be the structure constant related to a fixed choice of basis for W. Let $A\times_C G$ denote the extension of A by G defined above, and let $R = k[A\times_C G]$ be the Loop ring over $A\times_C G$. Let us define the vector subspace I of R generated by all elements of the form $I = \langle(\alpha, g) - \alpha(1, g) : \alpha \in A, g \in G\rangle$. Then $I \subset R$ is a bilateral ideal and R/I is a k-algebra isomorphic to W.*

Proof. It is an elementary fact that I is a bilateral ideal of R.

Now, for $(\alpha, a) \in A\times_C G$ let us define $\varphi : A\times_C G \to W^*$ as the function which sends (α, a) to αv_a,where W^* denotes the invertible elements of W. Clearly, $\varphi((\alpha, a)(\beta, b)) = \varphi(\alpha, a)\varphi(\beta, b)$, and consequently φ can be extended to a k-linear map (that we will denote by the same letter) $\varphi : R \to W$. An easy computation shows that $I \subset Ker(\varphi)$, and consequently φ descends to the quotient $\varphi : R/I \to W$, sending the class of $\sum \lambda_{(\alpha,a)}(\alpha, a)$ into $\sum \lambda_{(\alpha,a)}\alpha v_a$. An inverse map can be defined explicitly by extending linearly the map $\phi : W \to R/I$ that sends each element v_a to the class of $(1, a)$. ϕ is indeed a homomorphism of k algebras: ("$-$" denotes the class of an element)

$$\begin{aligned}
\phi(\sum_{a\in G}\lambda_a v_a \sum_{b\in G}\lambda_b v_b) &= \phi(\sum_{a,b\in G}\lambda_a\lambda_b C(a,b)v_{ab})\\
&= \phi(\sum_{g\in G}(\sum_{a+b=g} C(a,b)\lambda_a\lambda_b)v_{ab})\\
&= \sum_{g\in G}\left(\sum_{a+b=g} C(a,b)\lambda_a\lambda_b\right)(1,g)^- \qquad (3.1)
\end{aligned}$$

A similar computation shows that (3.1) is equal to $\phi(\sum\limits_{a\in G}\lambda_a v_a)\phi(\sum\limits_{b\in G}\lambda_b v_b)$. □

4. Associative G-graded algebras over $\mathbb{C}$

Standard techniques of group representation theory allows us to completely classify all G-graded twisted associative $\mathbb{C}$-algebras. For this, let us fix a basis B and let $C : G\times G \to A \subset \mathbb{C}^*$ be the structure constant of W with respect to B. As observed above, whenever W is associative $E_W = A\times_C G$ is not only a loop, but a group. Let $R = \mathbb{C}[A\times_C G]$ be its associated group ring. It is a standard result that R is the regular representation of E_W, and that if R decomposes into irreducible representations $R = V_1^{a_1} \oplus\cdots\oplus V_r^{a_r}$, with $V_i \neq V_j$, then the exponents a_i can be computed in terms of the characters χ_R and χ_{V_i} as

$$a_i = \langle\chi_R, \chi_{V_i}\rangle = \frac{1}{|A\times_C G|}\sum_{g\in G}\overline{\chi_R(g)}\chi_{V_i}(g) = \dim V_i.$$

(See [7], [8]). Hence, the homomorphism $\varphi : \mathbb{C}[A\times_C G] \to \oplus_{i=1}^r Hom_{\mathbb{C}}(V_i, V_i)$ that send each element $s \in \mathbb{C}[A\times_C G]$ into $(\psi_i)_{i=1,\ldots,r}$, where $\psi_i : V_i \to V_i$

denotes multiplication by s, is in fact an isomorphism of $\mathbb{C}$-algebras (no grading involved) [8]. With notation as in Theorem 1:

Theorem 2. *There is an isomorphism (not necessarily graded) of $\mathbb{C}$-algebras*

$$\begin{aligned} \varphi : \mathbb{C}\left[A \times_C G\right]/I &\rightarrow \oplus_{i=1}^{r} Hom_{\mathbb{C}}(V_i, V_i)/J_i \\ [s] &\mapsto (\psi_i)_{i=1,\dots,r} \end{aligned}$$

where $J = \varphi(I)$ is isomorphic to a product $J_1 \times \cdots \times J_n$, of bilateral ideals $J_i \subset Hom_{\mathbb{C}}(V_i, V_i)$.

Proof. The ideal J decomposes as a product of bilateral ideals $J = J_1 \times \cdots \times J_r$, with $J_i \subset Hom_{\mathbb{C}}(V_i, V_i)$. Hence,

$$W \cong R/I = \oplus_{i=1}^{r} Hom_{\mathbb{C}}(V_i, V_i)/J_i.$$

But each one of the algebras $Hom_{\mathbb{C}}(V_i, V_i)$ is simple and consequently $J_i = 0$ or $J_i = Hom_{\mathbb{C}}(V_i, V_i)$. Thus, $W \cong \oplus_i Hom_{\mathbb{C}}(V_i, V_i)$ where the sum occurs only for those i such that $J_i = (0)$. □

Corollary 1. *Let W be a G-graded twisted associative $\mathbb{C}$-algebra. Then W is isomorphic as a $\mathbb{C}$-algebra to a finite product of matrix algebras $Mat_{n_i \times n_i}(\mathbb{C})$, where $n_i = \dim V_i$.The algebra W is commutative if and only if $n_i = \dim V_i = 1$, and therefore, if and only if $W \simeq \mathbb{C} \times \cdots \times \mathbb{C}$.*

4.1. **Associative G-graded algebras over $\mathbb{R}$.** For the real case we proceed in a similar manner as in the last section. For any group G, the group ring $\mathbb{R}[G]$ is isomorphic to a finite direct sum of $\mathbb{R}$-algebras of the form $Hom_{D_i}(V_i, V_i)$ where D_i is a division algebra over the reals. Moreover, $D_i = Hom_{\mathbb{R}}(V_i, V_i)^G$ (see [8]). This immediately yields the following:

Theorem 3. *Let $W = \oplus_{g \in G} W_g$ be a G-graded, twisted, associative $\mathbb{R}$-algebra. Then W is isomorphic to a direct sum $\oplus_{i=1}^{r} Hom_{D_i}(V_i, V_i)$, where D_i denotes one of the division rings $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the quaternions.*

Proof. We already know that $W \simeq \mathbb{R}[A \times_C G]/I$, for some bilateral ideal I. From the remark above (4.1) it follows that

$$\mathbb{R}[A \times_C G]/I \simeq \oplus_{i=1}^{n} Hom_{D_i}(V_i, V_i)/J_i,$$

for bilateral ideals J_i of $Hom_{D_i}(V_i, V_i)$, where $D_i = Hom_{\mathbb{R}}(V_i, V_i)^{A \times_C G}$ is a division (associative) algebra over the reals. But it is a well known fact that D_i must equal to one of the algebras $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Thus, the result follows. □

5. Graded Morphisms and group Cohomology

In this section we study the problem of determining when two G-graded twisted k-algebras are isomorphic under a graded isomorphism. The following theorem provides necessary and sufficient conditions for two algebras to be graded-isomorphic in terms of the second group cohomology $H^2(G, k^*)$. For the basic notions about group cohomology, the reader may consult [1], [2].

Theorem 4. *Let $V = \oplus_{g\in G} V_g$ and $W = \oplus_{g\in G} W_g$ two G-graded k-algebras. Let us fix bases B_1 and B_2 for V and W, respectively, and let C_1 and C_2 be the associated structure constants. Then V is isomorphic to W if and only if the function $C_1C_2^{-1}$ is in the kernel of $d^2 : C^2(G,k^*) \to C^3(G,k^*)$ and the class $[C_1C_2^{-1}]$ is trivial in $H^2(G,k^*)$.*

Proof. Let us suppose that V and W are isomorphic as k-algebras under a grading preserving isomorphism $\phi : V \to W$. This implies that there exists $\varphi : G \to k^*$ which sends each vector v_g into $\varphi(g)w_g$. But ϕ a homomorphism implies that $\phi(C_1(a,b)v_{a+b}) = \varphi(a)w_a\varphi(b)w_b$, and consequently, $C_1(a,b)\varphi(ab)w_{a+b} = \varphi(a)\varphi(b)C_2(a,b)w_{a+b}$. From this we obtain

$$C_1(a,b)C_2^{-1}(a,b) = \varphi(a)\varphi(ab)^{-1}\varphi(b). \tag{5.1}$$

Notice that $d^1\varphi(a,b) = \varphi(b)\varphi^{-1}(ab)\varphi(a)$, and therefore $C_1C_2^{-1}$ belongs to the image of $d^1 : C^1(G,k^*) \to C^2(G,k^*)$. Thus, $d^2(C_1C_2^{-1}) = 1$, and $[C_1C_2^{-1}] = 1$ in $H^2(G,k^*)$.

Reciprocally, if $d^2(C_1C_2^{-1}) = 1$, and $[C_1C_2^{-1}] = 1$ in $H^2(G,k^*)$, then there exists $\varphi : G \to k^*$ such that $d^1\varphi = C_1C_2^{-1}$ and consequently equation (5.1) holds. It then follows that the function $\phi : V \to W$ defined on the basis B_1 as $\phi(v_g) = \varphi(g)w_g$ is a homomorphism of k-algebras, which is injective, since so it is ϕ. But V are W are k-vector spaces of the same dimension (equal to $|G|$). Hence, ϕ is an isomorphism. □

Remark 2. If V and W are associative, then $d^2C_1 = d^2C_2 = 1$, since both these terms are equal to the associativity function in (2.1). In this case, the class of each C_i is an element of $H^2(G,k^*)$ and the condition $[C_1C_2^{-1}] = 1$ is equivalent to $[C_1] = [C_2]$ in $H^2(G,k^*)$.

Corollary 2. *Let $W = \oplus_{g\in G} W_g$ be a G-graded k-algebra, and let be B and B' be bases for W, with associated constant structures C and C'. Let r and r' be the corresponding functions as defined above. Then $r = r'$. In other words, the associativity function of W does not depend on any chosen basis.*

Proof. The identity isomorphism $I : W \to W$ is trivially graded. By the previous theorem, $[C'C^{-1}] = 1$ and consequently $C'C^{-1} \in im(d^1)$. Hence, $d^2(C'C^{-1}) = 1$, and therefore $r' = d^2(C') = d^2(C) = r$. □

Let us notice now that if C_1 and C_2 take values in a subdomain $A \subset k^*$, then $C_1C_2^{-1} \in C^2(G,A)$. Hence, if $d^2(C_1C_2^{-1}) = 1$, it makes sense to talk about the class $[C_1C_2^{-1}] \in H^2(G,A)$. The following theorem gives a criterion in terms of $H^2(G,A)$ to determine when V and W are isomorphic. This may be useful in many cases where A is a finite subgroup of k^*.

Theorem 5. *$\phi : V \to W$ is a (graded) isomorphism if and only if $d^2(C_1C_2^{-1}) = 1$, and $[C_1C_2^{-1}] \in \ker(i_2)$, where $i_2 : H^2(G,A) \to H^2(G,k^*)$ denotes the homomorphism in cohomology induced by the inclusion $i : A \to k^*$.*

Proof. The short exact sequence of groups

$$1 \to A \xrightarrow{i} k \xrightarrow{\pi} k^*/A \to 1.$$

induces an exact sequence of complexes

$$1 \to C^\bullet(G,A) \xrightarrow{i_\bullet} C^\bullet(G,k^*) \xrightarrow{\pi_\bullet} C^\bullet(G,k^*/A) \to 1, \qquad ((*))$$

where we may identify the quotient $C^\bullet(G,k^*)/C^\bullet(G,A)$ with $C^\bullet(G,k^*/A)$ via the isomorphism that sends the class of $h : G \to k^*$ into $\pi \circ h$. By Theorem 4, V and W are graded isomorphic, if and only if $d^2(C_1C_2^{-1}) = 1$, and $[C_1C_2^{-1}] = 1$ in $H^2(G,k^*)$. But looking at the long exact sequence for cohomology

$$\cdots \to H^1(G,k^*) \xrightarrow{\pi_1} H^1(G,k^*/A) \xrightarrow{\delta} H^2(G,A) \xrightarrow{i_2} H^2(G,k^*) \to \cdots$$

we see that this occurs precisely when $i_2([C_1C_2^{-1}]) = 1$. □

Example 1. If G denotes a cyclic group of order n, then it is well know that $H^2(G,C^*) = \mathbb{C}^*/(\mathbb{C}^*)^n = \{1\}$. Hence, if V and W are G-graded associative algebras with structure constants given by $C_1, C_2 : G \times G \to A \subset C^*$, then $[C_1][C_2]^{-1} = 1$ and consequently they are isomorphic. It readily follows that $C[t]/(t^n - 1) = \oplus_{r=0}^{n-1} Ct^r$ is a representative of the unique isomorphism class.

Remark 3. If $k = R$, then $H^2(G,R^*) = \{1\}$, if n is odd, and it is equal to $\{1,-1\}$, if n is even. In the first case, there exists a unique real associative algebra $R[t]/(t^n - 1)$. In the second case, there are exactly two algebras, given by $R[t]/(t^n - 1)$, and $R[t]/(t^n + 1)$. On the other hand, if V and W have structure constants $C_1, C_2 : G \times G \to A$, where $|A|$ and $|G|$ are relatively prime integers, then $H^2(G,A) = \{1\}$. From the previous theorem, if $d^2(C_1C_2^{-1}) = 1$ then V and W are isomorphic as graded algebras. In the general case $d^2(C_1C_2^{-1}) = d^2(C_1)d^2(C_2)^{-1} = r_1r_2^{-1}$, where r_i is the associativity function of C_i. Thus, $r_1 = r_2$ if and only if V and W are isomorphic.

This proves the following.

Theorem 6. *Let $W^1 = \oplus_{g\in G} W_g^1$, and $W^2 = \oplus_{g\in G} W_g^2$, be G-graded k-algebras over a finite group G. Let $r_1,r_2 : G^3 \to A$ be the corresponding associativity functions. If $|A|$ and $|G|$ are relatively prime integers, then W^1 and W^2 are isomorphic as graded algebras if and only if $r_1 = r_2$. In particular, if W^1 and W^2 are associative, and if $|A|$ and $|G|$ are relatively prime integers, then W^1 and W^2 are isomorphic as graded algebras.*

6. Some classification results in the nonassociative case

In this section we shall give a complete classification under graded isomorphisms of all G-graded twisted algebras, when G is a finite cyclic group, and under the condition that their associative function satisfies the following symmetric condition: $r(a,b,c) = r(b,a,c)$, for all $a,b,c \in G$.

We start with a general discussion. Let G be any abelian group G, and let $W = \oplus_{g\in G} W_g$ be a G-graded twisted k-algebra. Let us fix a basis $B = \{v_g : g \in G\}$, and denote by $T_g : W \to W$ the k-linear map defined by multiplying (on the left) by v_g. If a and b are arbitrary elements of G, then for any $v_g \in B$

$$\begin{aligned} T_a(T_b(v_g)) &= T_a(v_b v_g) \\ &= v_a(v_b v_g) \\ &= r(a,b,g)(v_a v_b)v_g \\ &= r(a,b,g)C(a,b)v_{ab}v_g \\ &= r(a,b,g)C(a,b)T_{ab}(v_g), \end{aligned}$$

and consequently,

$$T_{ab}(v_g) = r(a,b,g)^{-1}C(a,b)^{-1}T_a(T_b(v_g)). \tag{6.1}$$

Similarly,

$$T_{ba}(v_g) = r(b,a,g)^{-1}C(b,a)^{-1}T_b(T_a(v_g)). \tag{6.2}$$

Since G is abelian, $T_{ab} = T_{ba}$. Therefore,

$$T_a(T_b(v_g)) = r(a,b,g)C(a,b)r(b,a,g)^{-1}C(b,a)^{-1}T_b(T_a(v_g)).$$

Using the symmetry condition (6) $r(a,b,g) = r(b,a,g)$, it follows that

$$T_a(T_b(v_g)) = q(a,b)T_b(T_a(v_g)).$$

Hence, for any $x \in W$ (not necessarily homogeneous)

$$T_a(T_b(x)) = q(a,b)T_b(T_a(x)) \tag{6.3}$$

In a similar way,

$$T_b(T_a(x)) = q(b,a)T_a(T_b(x)). \tag{6.4}$$

Let G be a cyclic group of order n. Fix any generator a of G. Let w_a be any non zero element in W_a. Define inductively $w_{a^k} = w_a \cdot w_{a^{k-1}}$, where $w_{a^0} = \alpha$, for some arbitrary fixed $\alpha \neq 0$ in $\mathbb{C}$. Define $v_{a^k} = \beta^k w_{a^k}$, where β denotes any primitive n-th root of unity of α^{-1}, ($\beta^n = 1/\alpha$). If $k = n$, we see that $v_{a^0} = \beta^n w_{a^0} = \beta^n\alpha = 1$, and

$$v_a v_{a^{k-1}} = \beta^k w_a w_{a^{k-1}} = \beta^k w_{a^k} = v_{a^k}.$$

The basis $B = \{1, v_a, \dots, v_{a^{n-1}}\}$ will be called the *standard basis* for W. In what follows, the structure constant $C_B(a^r, a^k)$ will be denoted by $C(a^r, a^k)$. Notice that $C(a, a^r) = 1$, for all $r = 0, \dots n-1$.

Since T_a sends v_{a^k} into $v_{a^{k+1}}$, the linear map T_a must permute the basis B cyclically, and consequently the minimal polynomial for T_a must be $Y^n - 1 = 0$. Hence, the eigenvalues of T_a are precisely the n-th complex roots of unity $\{\omega_1, \dots, \omega_n\}$. For each $1 \leq j \leq n$, let $z_j = \sum_{k=0}^{n-1} \omega_j^k v_{a^k}$.

If $t \equiv s \bmod n$, then $\omega_j^t v_{a^t} = \omega_j^s v_{a^s}$, since $t = s + nq$ implies

$$\omega_j^t v_{a^t} = \omega_j^{nq}\omega_j^s v_{a^s} = 1 \cdot \omega_j^s v_{a^s}.$$

Consequently, the sum $\sum_{k=0}^{n-1} \omega_j^k v_{a^k}$ can be also be written as $\sum_{k\in\mathbb{Z}_n} \omega_j^k v_{a^k}$. Now, ${\omega_j}^{-1}$ is an eigenvalue associated to z_j because

$$\begin{aligned} T_a(z_j) &= \sum_{k\in\mathbb{Z}_n} \omega_j^k v_a v_{a^k} = {\omega_j}^{-1} \sum_{k+1\in\mathbb{Z}_n} {\omega_j}^{k+1} v_{a^{k+1}} \\ &= {\omega_j}^{-1} \sum_{s\in\mathbb{Z}_n}^{n} \omega_j^s v_{a^s} = {\omega_j}^{-1} z_j. \end{aligned}$$

On the other hand, by (6.4)

$$\begin{aligned} T_b(T_a(z_j)) &= q(b,a) T_a(T_b(z_j)) \\ T_b({\omega_j}^{-1} z_j) &= q(b,a) T_a(T_b(z_j)) \\ {\omega_j}^{-1} T_b(z_j) &= q(b,a) T_a(T_b(z_j)) \\ q(a,b){\omega_j}^{-1} T_b(z_j) &= T_a(T_b(z_j)). \end{aligned}$$

Hence, $T_b(z_j)$ is a eigenvector of the eigenvalue $q(a,b){\omega_j}^{-1}$. Since T_a has n different eigenvalues, there must exist ω_i and z_i such that ${\omega_i}^{-1} = q(a,b){\omega_j}^{-1}$ and $T_b(z_j) = \eta_{b,j} z_i$, for some $\eta_{b,j} \neq 0$ en $\mathbb{C}$.

If we take $b = a^r$ we obtain

$$\begin{aligned} T_{a^r}(z_j) &= T_{a^r}\Big(\sum_{k\in\mathbb{Z}_n} \omega_j^k v_{a^k}\Big) = \sum_{k\in\mathbb{Z}_n} \omega_j^k C(a^r, a^k) v_{a^{r+k}} && (6.5) \\ &= \omega_j^{-r} \sum_{k+r\in\mathbb{Z}_n}^{n-1} \omega_j^{k+r} C(a^r, a^k) v_{a^{r+k}} && (6.6) \\ &= \omega_j^{-r} \sum_{s\in\mathbb{Z}_n} \omega_j^s C(a^r, a^{s-r}) v_{a^s}, && (6.7) \end{aligned}$$

$(s = k + r)$. On the other hand,

$$\begin{aligned} T_{a^r}(z_j) &= \eta_{a^r,j} z_i = \eta_{a^r,j} \sum_{s\in\mathbb{Z}_n} \omega_i^s v_{a^s} && (6.8) \\ &= \sum_{s\in\mathbb{Z}_n} \eta_{a^r,j} q(a^r, a)^s \omega_j^s v_{a^s}. && (6.9) \end{aligned}$$

Comparing coefficients in (6.7) and (6.8) we see that

$$\omega_j^{s-r} C(a^r, a^{s-r}) = \eta_{a^r,j} q(a^r, a)^s \omega_j^s$$

and consequently

$$C(a^r, a^{s-r}) = \eta_{a^r,j} q(a^r, a)^s {\omega_j}^r. \tag{6.10}$$

In particular, if $s = r$, we see that $1 = C(a^r, a^0) = \eta_{a^r,j} q(a^r, a)^r {\omega_j}^r$. Thus, $\eta_{a^r,j} = q(a, a^r)^r {\omega_j}^{-r}$. Substituting in (6.10) we get $C(a^r, a^{s-r}) = q(a, a^r)^r \omega_j^{-r} q(a^r, a)^s \omega_j^r$, and consequently $C(a^r, a^{s-r}) = q(a^r, a)^{s-r}$.

If we let $k = s - r$ we see that

$$C(a^r, a^k) = q(a^r, a)^k = C(a^r, a)^k C(a, a^r)^{-k} = C(a^r, a)^k,$$

(using that $C(a, a^r) = 1$). Thus,

$$\begin{aligned} (6.11) \quad r(a^r, a^s, a^t) &= C(a^s, a)^t C(a^{r+s}, a)^{-t} C(a^r, a)^{s+t} C(a^r, a)^{-s} \\ (6.12) \quad &= (C(a^s, a)C(a^r, a)C(a^{r+s}, a)^{-1})^t \end{aligned}$$

Let G be a cyclic group of order n and let $W = \oplus_{g\in G} W_g$, a G-graded twisted over $\mathbb{C}$. Let us assume that r is symmetric in the first two entries. If B denotes the standard basis for W, then W is completely determined by the function $f : G \to A$, given by $f(a^r) = C(a^r, a)$, where a is any fixed generator of G. Hence, there are precisely $|A|^{n-2}$ non isomorphic G-graded twisted algebras.

Example 2. Let $Z_4 = \{a, a^2, a^3, a^4 = 1\}$ and $W = \oplus_{r=0}^{3} W_{a^r}$, a C-algebra. Let $A = \{1, -1\}$. The minimal polynomial for T_a is $Y^4 - 1 = 0$, with eigenvalues $\omega_1 = -i, \omega_2 = -1, \omega_3 = i, \omega_4 = 1$. It is easy to see that the corresponding eigenvectors are

$$\begin{aligned} z_1 &= 1 + iv - v_2 - iv^3 \\ z_2 &= 1 - v + v^2 - v^3 \\ z_3 &= 1 - iv - v^2 + iv^3 \\ z_4 &= 1 + v + v^2 + v^3. \end{aligned}$$

Hence,

$$\begin{aligned} T_a(z_1) &= -iz_1 \\ T_a(z_2) &= -z_2 \\ T_a(z_3) &= iz_3 \\ T_a(z_4) &= z_4 \end{aligned}$$

Case 1 : Suppose $C(a^2, a) = -1$ and $C(a^3, a) = 1$. Therefore

$$\begin{aligned} T_a(T_{a^2}(z_1)) &= -T_{a^2}(T_a(z_1)) \\ T_a(T_{a^2}(z_1)) &= iT_{a^2}(z_1) \end{aligned}$$

Then, $T_{a^2}(z_1)$ is an eigenvector associated to the eigenvalue i; that is, $T_{a^2}(z_1) = \beta(a^2, 1)z_3$. Hence,

$$\beta(a^2, 1) = q(a, a^2)^2 \omega_1^{-2} = (-1)^2 (i)^{-2} = -1$$

and the structure constants are given by

$$\begin{aligned} C(a^2, a^2) &= q(a^2, a)^2 = 1 \\ C(a^2, a^3) &= q(a^2, a)^3 = -1 \\ C(a^2, a) &= q(a^2, a) = -1 \end{aligned}$$

Similarly, $T_{a^2}(z_2)$ is an eigenvector of the eigenvalue 1, i.e., $T_{a^2}(z_2) = \beta(a^2, 2)z_4$, with $\beta(a^2, 2) = 1$. And $T_{a^2}(z_3)$ is an eigenvector of the eigenvalue $-i$, i.e., $T_{a^2}(z_3) = \beta(a^2, 3)z_1$,with $\beta(a^2, 3) = -1$. Finally, $T_{a^2}(z_4)$ is an eigenvector of

the eigenvalue $-1 : T_{a^2}(z_4) = \beta(a^2, 4)z_2$, with $\beta(a^2, 4) = 1$. Now let $b = a^3$; since $q(a, a^3) = 1$, then

$$\begin{aligned} T_a(T_{a^3}(z_1)) &= T_{a^3}(T_a(z_1)) \\ T_a(T_{a^3}(z_1)) &= -iT_{a^3}(z_1) \end{aligned}$$

Then $T_{a^3}(z_1)$ is an eigenvector associated to $-i$, i.e., $T_{a^3}(z_1) = \beta(a^3, 1)z_1$. Thus,

$$\beta(a^3, 1) = q(a, a^3)^3\omega_1^{-3} = (1)^3(i)^{-3} = i,$$

and the structure constants are given by

$$\begin{aligned} C(a^3, a) &= q(a^3, a) = 1 \\ C(a^3, a^2) &= q(a^3, a)^2 = 1 \\ C(a^3, a^3) &= q(a^3, a)^3 = 1. \end{aligned}$$

Similarly, $T_{a^3}(z_2) = \beta(a^3, 2)z_2$, with $\beta(a^3, 2) = -1$; $T_{a^3}(z_3) = \beta(a^3, 3)z_3$, with $\beta(a^3, 3) = -i$. And $T_{a^3}(z_4) = \beta(a^3, 4)z_3$, with $\beta(a^3, 4) = 1$.

In the same way, the remaining three cases are analyzed: Case 2, when $q(a, a^2) = -1$ and $q(a, a^3) = -1$. Case 3, when $q(a, a^2) = 1$ and $q(a, a^3) = 1$. Case 4: $q(a, a^2) = 1$ and $q(a, a^3) = -1$.

Consequently, there are $|A|^{n-2} = (2)^2$ graded twisted C-algebras over $\mathbb{Z}_4$.

Putting all together we obtain

Theorem 7. *Let W_1, W_2 be $\mathbb{Z}_n$-graded algebras over $k = \mathbb{C}$ or $\mathbb{R}$ with structure constants C_1 and C_2 corresponding to the standard basis B_1 and B_2, respectively, and associativity function r which is symmetric in the first two components. Then $W_1 \cong W_2$ as graded k-algebras if and only if $C_1(a^r, a) = C_2(a^r, a)$, for all $1 \le r \le n$.*

Proof. By a previous theorem we know that $W_1 \cong W_2$ if and only if $d^2(C_1C_2^{-1}) = 1$; Or equivalently, if and only if $d^2C_1 = d^2C_2$. That is, if and only if, $r_1 = r_2$. Now, if $r_1 = r_2$, then equation (6.12) is true for all t, in particular if $t = 1$. Now, define $f_i : G \to A$ by $f_i(a^r) = C_i(a^r, a)$. Applying d^1 we obtain

$$d^1 f_i(a^r, a^s) = f_i(a^s)f_i(a^{r+s})^{-1}f_i(a^r).$$

Hence, $r_1 = r_2$ if and only if $d^1 f_1 = d^1 f_2$; Or equivalently, iff $(f_1f_2^{-1}) \in Ker(d^1)$. But

$$\begin{aligned} Ker(d^1) &= H^1(G, A) = \{h : G \to A : h(a)h(b)h(ab)^{-1} = 1\} \\ &= \{h : G \to A : h \text{ is a group homomorphism}\}. \end{aligned}$$

Therefore, $f_1 = f_2h$ for some group homomorphism h. This implies that

$$C_1(a^r, a) = h(a)C_2(a^r, a), \text{ for all } 1 \le r \le n - 1.$$

If $r = 1$, we see that $1 = C_1(a, a) = h(a)C_2(a, a)$, and consequently $h(a) = 1$. Summarizing, we obtain $C_1(a^r, a) = C_2(a^r, a)$, for $1 \le r \le n - 1$. $\square$

6.1. **Classification in the real case.** With notation as above: Let $W = \oplus_{g\in G} W_g$ be a G-graded twisted $\mathbb{R}$- algebra over a cyclic group G of order n. Let us fix $a \in G$ a generator for this group, and let w_a be a nonzero element in W_a. For $1 \le k \le n-1$, we define inductively, $w_{a^k} = w_a \cdot w_{a^{k-1}}$, with $w_{a^0} = \alpha$, for some fixed $\alpha \ne 0$ in $\mathbb{R}$. Let us distinguish two cases:

(1) n is odd, or n is even and $\alpha > 0$. In this case, we define $v_{a^k} = \beta^k w_{a^k}$ for each $0 < k \le n$, where β is any n-root of α^{-1}. Clearly, if $k = n$, then $v_{a^0} = \beta^n w_{a^0} = \beta^n \alpha = 1$. Moreover, $v_a v_{a^{k-1}} = \beta^k w_a w_{a^{k-1}} = \beta^k w_{a^k} = v_{a^k}$.

(2) n is even and $\alpha < 0$. Define $v_{a^0} = 1$, and for $0 < k < n$, define $v_{a^k} = \beta^k w_{a^k}$, where $\beta^n = -1/\alpha$. Hence, for all $k < n$

$$v_a v_{a^{k-1}} = \beta^k w_a w_{a^{k-1}} = \beta^k w_{a^k} = v_{a^k}.$$

And for $k = n$, it holds that $v_a v_{a^{n-1}} = \beta^n w_a w_{a^{n-1}} = -1/\alpha \cdot \alpha = -1$.

In any of the previous two cases, we will call $B = \{1, v_a, \ldots, v_{a^{n-1}}\}$ the *standard basis* for W. Again, for brevity, we will omit the subindex B in the notation of the structure constant corresponding to this basis.

Now we notice that in the first case $C(a, a^r) = 1$, for all r; in the second case, $C(a, a^r) = 1$, if $0 \le r < n-1$, and $C(a, a^{n-1}) = -1$. This can be written in a short manner as $C(a^r, a) = (-1)^{\delta(n-1,r)}$, where δ is the delta of Kronecker.

The classification problem in the first case is identical as in the complex case. Hence, we will restrict to the second case, i.e., when n is even and $\alpha < 0$.

Since T_a sends v_{a^k} into $v_{a^{k+1}}$, the linear map T_a permutes cyclically the basis B, with the exception that it sends $v_{a^{n-1}}$ to $-v_{a^0} = -1$. Hence, the minimal polynomial for T_a must be $Y^n + 1 = 0$. From this, the eigenvalues of T_a are the set of all complex n-roots of -1, that we will denote by $\{\omega_1, \ldots, \omega_n\}$. For each $1 \le j \le n$, we define $z_j = \sum_{k=0}^{n-1} \omega_j^k v_{a^k}$. Let us see the effect of multiplying v_{a^r} on z_j :

$$\begin{aligned} v_{a^r} \sum_{k=0}^{n-1} \omega_j^k v_{a^k} &= \sum_{k=0}^{n-1} \omega_j^k C(a^r, a^k) v_{a^{k+r}} \\ &= \omega_j^{-r} \sum_{k=0}^{n-r-1} \omega_j^{k+r} C(a^r, a^k) v_{a^{k+r}} \\ &\quad + \omega_j^{-r} \sum_{k=n-r}^{n-1} \omega_j^{k+r} C(a^r, a^k) v_{a^{k+r}} \end{aligned} \tag{6.13}$$

The first sum of (6.13) may be rewritten as $\omega_j^{-r} \sum_{s=r}^{n-1} \omega_j^s C(a^r, a^{s-r}) v_{a^s}$ (with $s = k + r$). If $s = k - (n-r)$, we see that the second sum in (6.13) can be written as

$$(-1)\omega_j^{-r} \sum_{s=0}^{r-1} \omega_j^s C(a^r, a^{s+n-r}) v_{a^{n+s}},$$

which is equal to $(-1)\omega_j^{-r}\sum_{s=0}^{r-1}\omega_j^s C(a^r,a^{n+s-r})v_{a^s}$, since $\omega_j^n=-1$. Hence,

$$v_{a^r}z_j=(-1)\omega_j^{-r}\sum_{s=0}^{r-1}\omega_j^s C(a^r,a^{n+s-r})v_{a^s}+\omega_j^{-r}\sum_{s=r}^{n-1}\omega_j^s C(a^r,a^{s-r})v_{a^s} \tag{6.14}$$

Now, ω_j^{-1} is an eigenvalue of z_j. Indeed, if we let $r=1$ in the equation above,

$$\begin{aligned}
T_a(z_j) &= v_a z_j=(-1)\omega_j^{-1}C(a^1,a^{n-1})v_{a^0}\\
&\quad +\omega_j^{-1}\sum_{s=1}^{n-1}\omega_j^s C(a,a^{s-1})v_{a^s}\\
&= \omega_j^{-1}v_{a^0}+\omega_j^{-1}\sum_{s=1}^{n-1}\omega_j^s C(a,a^{s-1})v_{a^s}=\omega_j^{-1}z_j.
\end{aligned}$$

since $C(a^1,a^{s-1})=1$, when $s=1,\dots,n-1$.

On the other hand, form (6.4) is obtained:

$$\begin{aligned}
T_b(T_a(z_j)) &= q(b,a)T_a(T_b(z_j))\\
T_b(\omega_j^{-1}z_j) &= q(b,a)T_a(T_b(z_j))\\
\omega_j^{-1}T_b(z_j) &= q(b,a)T_a(T_b(z_j))\\
q(a,b)\omega_j^{-1}T_b(z_j) &= T_a(T_b(z_j)).
\end{aligned}$$

Hence, $T_b(z_j)$ is an eigenvector associated to the eigenvalue $q(a,b)\omega_j^{-1}$.But since T_a has n different eigenvalues there must be ω_i and z_i such that $\omega_i^{-1}=q(a,b)\omega_j^{-1}$, and $T_b(z_j)=\sigma_{b,j}z_i$, for some real number $\sigma_{b,j}\neq 0$. Letting $b=a^r$ we get

$$\begin{aligned}
T_{a^r}(z_j) &= \sigma_{a^r,j}z_i=\sigma_{a^r,j}\sum_{k=0}^{n-1}\omega_i^k v_{a^k}\\
&= \sigma_{a^r,j}\sum_{s=0}^{n-1}q(a^r,a)^s\omega_j^s v_{a^s} \qquad (6.15)
\end{aligned}$$

Comparing coefficients in (6.14) and (6.15) we see that

$$(-1)\omega_j^{-r}C(a^r,a^{n+s-r})=\sigma_{a^r,j}q(a^r,a)^s,\ \text{if } 0\le s\le r-1, \tag{6.16}$$

and

$$\omega_j^{-r}C(a^r,a^{s-r})=\sigma_{a^r,j}q(a^r,a)^s,\quad \text{if } r\le s\le n-1. \tag{6.17}$$

Letting $s=r$ in (6.17), we obtain $\sigma_{a^r,j}=q(a^r,a)^{-r}\omega_j^{-r}$. Substituting this values in (6.16) we finally get

$$(-1)C(a^r,a^{n+s-r})=q(a,a^r)^{r-s},\ \text{if } 0\le s\le r-1.$$

Letting $k=n+s-r$ we see that

$$C(a^r,a^k)=(-1)q(a,a^r)^{n-k}=(-1)q(a^r,a)^k,\ \text{if } n-r\le k\le n-1, \tag{6.18}$$

since $q(a^r,a)^n = (\omega_j^{-1}\omega_i)^n = 1$. In a similar manner

$$C(a^r,a^k) = q(a^r,a)^k, \text{ if } 0 \le k \le n-r-1. \tag{6.19}$$

But we know that $C(a,a^r) = (-1)^{\delta(n-1,r)}$, from which

$$q(a^r,a) = C(a^r,a)C(a,a^r)^{-1} = (-1)^{\delta(n-1,r)}C(a^r,a).$$

The equalities (6.18 and 6.19) may be written in a simpler form as

$$\begin{aligned} C(a^r,a^k) &= (-1)^{k\delta(n-1,r)}C(a^r,a)^k, && \text{if } 0 \le k \le n-r-1. \\ C(a^r,a^k) &= (-1)^{k\delta(n-1,r)+1}C(a^r,a)^k, && \text{if } n-r \le k \le n-1. \end{aligned}$$

From this discussion we deduce the following theorem.

Theorem 8. *Let $G = \langle a^r : r = 0, \dots, 2m-1\rangle$ be a cyclic group of order $n = 2m$. Let $W = \oplus_{r=0}^{n-1} W_{a^r}$ be a G-graded twisted real algebra with symmetric associativity function r. Let B denote the standard basis for W. Then W is completely determined by the function $f : G \to A$, $f(a^r) = C_B(a^r,a)$. Thus, there are exactly $2\,|A|^{n-2}$ G-graded twisted non isomorphic algebras over the reals.*

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